Supersymmetric Probes in a Rotating 5D Attractor

Wei Li and Andrew Strominger

Jefferson Physical Laboratory, Harvard University, Cambridge MA 02138, USA

Abstract

Supersymmetric zero-brane and one-brane probes in the squashed $AdS_2 \times S^3$ near-horizon geometry of the BMPV black hole are studied. Supersymmetric zero-brane probes stabilized by orbital angular momentum on the S^3 are found and shown to saturate a BPS bound. We also find supersymmetric one-brane probes which have momentum and winding around a $U(1)_L \times U(1)_R$ torus in the S^3 and in some cases are static.

Contents

1 Introduction

The near-horizon attractor geometry of a BPS black hole has twice as many supersymmetries as the full asymptotically flat solution. In four dimensions, such geometries admit BPS probe configurations which preserve only near-horizon supersymmetries, and break all of the supersymmetries of the original asymptotically flat solution [1]. A novel feature of these configurations is that branes and anti-branes antipodally located on the S^2 preserve the same supersymmetries. Quantization of these classical configurations leads to lowest Landau levels which tile the black hole horizon [2]. In some cases the degeneracies saturate the Bekenstein-Hawking black hole entropy [3]. Furthermore, an appropriate expansion of the black hole partition function in a dilute gas of these states [4] yields a derivation of the OSV relation [5].

These interesting 4D phenomena should all have closely related 5D cousins [6]. With this in mind, the present paper extends the 4D classical BPS probe analysis of [1] to five dimensions. The 5D problem is considerably enriched by the fact that 5D BMPV BPS black holes can carry angular momentum J and have a $U(1)_L \times SU(2)_R$ rotational isometry group [7]. BPS zero-brane probes that orbit the S^3 are found using a κ-symmetry analysis. Their location in AdS_2 depends on the azimuthal angle on S^3, the background rotation J, and the angular momentum of the probe. For one-branes, we find BPS configurations with momentum and winding around a torus generated by a $U(1)_L \times U(1)_R$ rotational subgroup.[1] A one-brane in five dimensions can carry the magnetic charge dual to the electric charge supporting the BMPV black hole. Interestingly, we find that this allows for static BPS "black ring" configurations, where the angular momentum required for saturation of the BPS bound is carried by the gauge field.

[1]Inclusion of these states in the partition function of [4] could lead to non-factorizing corrections to the OSV relation.

2 Review of the BMPV black hole

The 5D $\mathcal{N}=2$ supersymmetric rotating black hole arises from M2-branes wrapping holomorphic curves of a Calabi-Yau threefold X. It is characterized by electric charges q_A, $A=1,2,..b_2(X)$, and the angular momentum J in $SU(2)_{\text{left}}$. The metric is [7]

$$\begin{aligned} ds^2 &= -\left(1+\frac{Q}{r^2}\right)^{-2}\left[dt+\frac{J}{2r^2}\sigma_3\right]^2+\left(1+\frac{Q}{r^2}\right)\left(dr^2+r^2d\Omega_3^2\right), \qquad (1)\\ d\Omega_3^2 &= \frac{1}{4}\left[d\theta^2+d\phi^2+d\psi^2+2\cos\theta d\psi d\phi\right]=\frac{1}{4}\sum_{i=1}^{3}(\sigma_i)^2, \qquad (2)\end{aligned}$$

where the ranges of the angular parameters are

$$\theta\in[0,\pi], \qquad \phi\in[0,2\pi], \qquad \psi\in[0,4\pi]. \tag{3}$$

σ_i are the right-invariant one-forms:[2]

$$\begin{aligned}\sigma_1 &= -\sin\psi d\theta+\cos\psi\sin\theta d\phi,\\ \sigma_2 &= \cos\psi d\theta+\sin\psi\sin\theta d\phi,\\ \sigma_3 &= d\psi+\cos\theta d\phi,\end{aligned} \tag{4}$$

and we choose Planck units $l_5=(\frac{4G_5}{\pi})^{1/3}=1$. The graviphoton charge Q is determined via the equations

$$Q^{\frac{3}{2}}=D_{ABC}y^Ay^By^C, \tag{5}$$

$$q_A=3D_{ABC}y^By^C, \tag{6}$$

with D_{ABC} the intersection form on X.

The near-horizon limit ($r\to 0$) of the metric is

$$ds^2=-\left[\frac{r^2}{Q}dt+\frac{J}{2Q}\sigma_3\right]^2+Q\frac{dr^2}{r^2}+Qd\Omega_3^2. \tag{7}$$

Rescaling t to absorb Q, defining $\sin^2 B=\frac{J^2}{Q^3}$ and $r^2=1/\sigma$, we obtain the metric in Poincaré coordinates:

$$ds^2=\frac{Q}{4}\left[-(\frac{dt}{\sigma}+\sin B\sigma_3)^2+\frac{d\sigma^2}{\sigma^2}+\sigma_1^2+\sigma_2^2+\sigma_3^2\right]. \tag{8}$$

[2]The $SU(2)$ rotation matrix is parameterized as:

$$e^{i\frac{\sigma_z}{2}\psi}e^{i\frac{\sigma_y}{2}\theta}e^{i\frac{\sigma_z}{2}\phi}=\begin{pmatrix}\cos\frac{\theta}{2}e^{i(\psi+\phi)/2} & \sin\frac{\theta}{2}e^{i(\psi-\phi)/2}\\ -\sin\frac{\theta}{2}e^{-i(\psi-\phi)/2} & \cos\frac{\theta}{2}e^{-i(\psi+\phi)/2}\end{pmatrix}.$$

The graviphoton field strength in these coordinates is

$$F_{[2]} = dA_{[1]}, \qquad A_{[1]} = \frac{\sqrt{Q}}{2}[\frac{1}{\sigma}dt + \sin B\sigma_3]. \tag{9}$$

We will also be using the metric in the global coordinates $(\tau, \chi, \theta, \phi, \psi)$:[3]

$$ds^2 = \frac{Q}{4}\left[-\cosh^2\chi d\tau^2 + d\chi^2 + (\sin B \sinh\chi d\tau - \cos B\sigma_3)^2 + \sigma_1^2 + \sigma_2^2\right], \tag{10}$$

in which

$$A_{[1]} = \frac{\sqrt{Q}}{2}[\cos B \sinh\chi d\tau + \sin B\sigma_3]. \tag{11}$$

The near horizon geometry of the BMPV black hole is a kind of squashed $AdS_2 \times S^3$. The near-horizon isometry supergroup is $SU(1,1|2) \times U(1)_{\text{left}}$, where the bosonic subgroup of $SU(1,1|2)$ is $SU(1,1) \times SU(2)_{\text{right}}$ [10]. When $J = 0$, $U(1)_{\text{left}}$ is promoted to $SU(2)_{\text{left}}$ and the full $SO(4) \cong SU(2)_{\text{right}} \times SU(2)_{\text{left}}$ rotational invariance is restored. The unbroken rotational symmetries for $J \neq 0$ are generated by the Killing vectors

$$\xi_3^L = \partial_\psi \tag{12}$$

and

$$\begin{aligned} \xi_1^R &= \sin\phi\partial_\theta + \cos\phi(\cot\theta\partial_\phi - \csc\theta\partial_\psi), \\ \xi_2^R &= \cos\phi\partial_\theta - \sin\phi(\cot\theta\partial_\phi - \csc\theta\partial_\psi), \\ \xi_3^R &= \partial_\phi. \end{aligned} \tag{13}$$

The supersymmetries arise from Killing spinors ϵ which are the solutions of the equation

$$\left[d + \frac{1}{4}\omega_{ab}\Gamma^{ab} + \frac{i}{8}\left(e^a\Gamma^{bc}\Gamma_a F_{bc} - 4e^a\Gamma^b F_{ab}\right)\right]\epsilon = 0 \tag{14}$$

[3]The coordinate transformation between the global coordinates and Poincaré ones is:

$$\begin{aligned} t &= \frac{\cos B \cosh\chi \sin\tau}{\cosh\chi\cos\tau + \sinh\chi}, \\ \sigma &= \frac{1}{\cosh\chi\cos\tau + \sinh\chi}, \\ \psi^{\text{Poincaré}} &= \psi^{\text{global}} + 2\tan B \tanh^{-1}(e^{-\chi}\tan\frac{\tau}{2}). \end{aligned}$$

To solve this in global coordinates we choose the vielbein

$$\begin{aligned}
e^0 &= \frac{\sqrt{Q}}{2}[\cosh(\sin B\cos B\psi)\cosh\chi d\tau + \sinh(\sin B\cos B\psi)d\chi], \\
e^1 &= \frac{\sqrt{Q}}{2}[\sinh(\sin B\cos B\psi)\cosh\chi d\tau + \cosh(\sin B\cos B\psi)d\chi], \\
e^2 &= \frac{\sqrt{Q}}{2}[-\sin(\cos^2 B\psi)d\theta + \cos(\cos^2 B\psi)\sin\theta d\phi], \\
e^3 &= \frac{\sqrt{Q}}{2}[\cos(\cos^2 B\psi)d\theta + \sin(\cos^2 B\psi)\sin\theta d\phi], \\
e^4 &= \frac{\sqrt{Q}}{2}[-\sin B\sinh\chi d\tau + \cos B\sigma_3].
\end{aligned} \tag{15}$$

The Killing spinors are then [8][9]

$$\begin{aligned}
\epsilon &= e^{\left[-\frac{1}{2}(\sin B\cos B\Gamma^{01}+\cos^2 B\Gamma^{23})\psi\right]} e^{\left[+\frac{1}{2}(\cos B\Gamma^{24}+i\sin B\Gamma^{2})\theta\right]} e^{\left[-\frac{1}{2}(\cos B\Gamma^{34}+i\sin B\Gamma^{3})\phi\right]} \\
&\quad e^{\left[+\frac{1}{2}(\sin B\Gamma^{04}-i\cos B\Gamma^{0})\chi\right]} e^{\left[-\frac{1}{2}(\sin B\Gamma^{14}-i\cos B\Gamma^{1})\tau\right]}\epsilon_0 \\
&\equiv S\epsilon_0,
\end{aligned} \tag{16}$$

where ϵ_0 is any spinor with constant components in the frame (15).

For Poincaré coordinates we choose the vielbein

$$e^0 = \frac{\sqrt{Q}}{2}[\frac{dt}{\sigma}+\sin B\sigma_3], \qquad e^1 = \frac{\sqrt{Q}}{2}\frac{d\sigma}{\sigma}, \qquad e^2 = \frac{\sqrt{Q}}{2}\sigma_1, \qquad e^3 = \frac{\sqrt{Q}}{2}\sigma_2, \qquad e^4 = \frac{\sqrt{Q}}{2}\sigma_3. \tag{17}$$

The Killing spinors are [10]

$$\epsilon^+ = \frac{1}{\sqrt{\sigma}}R(\theta,\phi,\psi)\epsilon_0^+, \tag{18}$$

$$\epsilon^- = \left[\sqrt{\sigma}(1-\sin B\Gamma^{04}) - \frac{t}{\sqrt{\sigma}}\Gamma^{01}\right]R(\theta,\phi,\psi)\epsilon_0^-, \tag{19}$$

where

$$\begin{aligned}
R(\theta,\phi,\psi) &= e^{-\frac{1}{2}\Gamma^{23}\psi}e^{\frac{1}{2}\Gamma^{24}\theta}e^{-\frac{1}{2}\Gamma^{23}\phi}, \\
i\Gamma^0\epsilon_0^\pm &= \pm\epsilon_0^\pm,
\end{aligned} \tag{20}$$

for constant $\epsilon_0^\pm$.

3 Supersymmetric probe configurations

In this section, we find classical brane trajectories which preserve some supersymmetries of the rotating attractor (7). The worldvolume action has a local κ-symmetry (parameterized

by κ) as well as a spacetime supersymmetry transformation (parameterized by ϵ) which acts nonlinearly. A spacetime supersymmetry is preserved if its action on the worldvolume fermions Θ can be compensated by a κ transformation [11][12]:

$$\delta_\epsilon \Theta + \delta_\kappa \Theta = \epsilon + (1+\Gamma)\kappa(\sigma) = 0, \tag{21}$$

where Γ is given in various cases analyzed below. This gives the condition

$$(1-\Gamma)\epsilon = 0, \tag{22}$$

which must be solved for both the Killing spinor and the probe trajectory.

3.1 Zero-brane probe

For the zero-brane the (bosonic part of the) κ-symmetry projection operator is

$$\Gamma = \frac{1}{\sqrt{h_{00}}}\tilde{\Gamma}_0, \tag{23}$$

where h and $\tilde{\Gamma}_0$ are the pull-backs of the metric and Dirac matrix onto the worldline of the zero-brane, respectively:

$$h_{00} = \partial_0 X^\mu \partial_0 X^\nu G_{\mu\nu}, \tag{24}$$

$$\tilde{\Gamma}_0 = \partial_0 X^\mu e^a_\mu \Gamma_a. \tag{25}$$

3.1.1 Global coordinates

First, let's look at the global coordinates. In the static gauge, where we set the worldvolume time σ^0 equal to the global time τ, the κ-symmetry operator is

$$\Gamma = \frac{1}{\sqrt{h_{00}}}\frac{dX^\mu}{d\tau} e^a_\mu \Gamma_a. \tag{26}$$

To solve for the classical trajectory of a supersymmetric zero-brane, we plug the Killing spinors (16) into the κ-symmetry condition (22) of the supersymmetric zero-brane. A zero-brane following a classical trajectory, given by $(\chi(\tau), \theta(\tau), \phi(\tau), \psi(\tau))$, is supersymmetric if, in the notation of (16),

$$\frac{1}{\sqrt{h_{00}}}\frac{dX^\mu}{d\tau} e^a_\mu S^{-1}\Gamma_a S \epsilon_0 = \epsilon_0, \tag{27}$$

for some constant ϵ_0, where $S = S(\chi, \tau, \theta, \phi, \psi)$. The explicit prefactors are

$$\begin{aligned}
S^{-1}e_0^a\Gamma_a S &= \frac{\sqrt{Q}}{2}[(\cosh\chi\cos\tau\cos^2 B + \sin\theta\cos\phi\sin^2 B)\Gamma^0 \\
&\quad + i\cosh\chi\sin\tau\cos B\Gamma^{01} - i\cos\theta\sin B\Gamma^{02} - i\sin\theta\sin\phi\sin B\Gamma^{03} \\
&\quad + i(\cosh\chi\cos\tau - \sin\theta\cos\phi)\sin B\cos B\Gamma^{04}], \\
S^{-1}e_1^a\Gamma_a S &= (-1)\frac{\sqrt{Q}}{2}[\sin\theta\cos\phi\cos\tau\Gamma^1 \\
&\quad - \sin\tau\sin B e^{\frac{1}{2}(\cos B\Gamma^{34}+i\sin B\Gamma^3)\phi} e^{-(\cos B\Gamma^{24}+i\sin B\Gamma^2)\theta} e^{\frac{1}{2}(\cos B\Gamma^{34}+i\sin B\Gamma^3)\phi}\Gamma^4 \\
&\quad - i\cos\tau\cos\theta\sin B\Gamma^{12} - i\cos\tau\sin\theta\sin\phi\sin B\Gamma^{13} + ie^{(\sin B\Gamma^{14}-i\cos B\Gamma^1)\tau}\sinh\chi\cos B\Gamma^{01} \\
&\quad + i(\cosh\chi - \sin\theta\cos\phi\cos\tau)\cos B(\sin B\Gamma^{14} - i\cos B\Gamma^1)], \\
S^{-1}e_2^a\Gamma_a S &= (-1)\frac{\sqrt{Q}}{2}[\cosh\chi\cos\tau\cos\phi\Gamma^3 - \cosh\chi\cos\tau\sin\phi\cos B\Gamma^4 \\
&\quad + e^{(\cos B\Gamma^{34}+i\sin B\Gamma^3)\phi}(+i\sinh\chi\cos B\Gamma^{03} - i\cosh\chi\sin\tau\cos B\Gamma^{13} + i\cos\theta\sin B\Gamma^{23}) \\
&\quad + i(\cosh\chi\cos\tau\cos\phi - \sin\theta)\sin B(\cos B\Gamma^{34} + i\sin B\Gamma^3)], \\
S^{-1}e_3^a\Gamma_a S &= (-1)\frac{\sqrt{Q}}{2}[(\cosh\chi\cos\tau\cos^2 B + \sin\theta\cos\phi\sin^2 B)\Gamma^2 \\
&\quad + i\sinh\chi\cos B\Gamma^{02} - i\cosh\chi\sin\tau\cos B\Gamma^{12} - i\sin\theta\sin\phi\sin B\Gamma^{23} \\
&\quad + i(\cosh\chi\cos\tau - \sin\theta\cos\phi)\sin B\cos B\Gamma^{24}], \\
S^{-1}e_4^a\Gamma_a S &= (-1)\frac{\sqrt{Q}}{2}\cos B e^{+\frac{1}{2}(\sin B\Gamma^{14}-i\cos B\Gamma^1)\tau} e^{-(\sin B\Gamma^{04}-i\cos B\Gamma^0)\chi} e^{+\frac{1}{2}(\sin B\Gamma^{14}-i\cos B\Gamma^1)\tau} \\
&\quad e^{+\frac{1}{2}(\cos B\Gamma^{34}+i\sin B\Gamma^3)\phi} e^{-(\cos B\Gamma^{24}+i\sin B\Gamma^2)\theta} e^{+\frac{1}{2}(\cos B\Gamma^{34}+i\sin B\Gamma^3)\phi}\Gamma^4.
\end{aligned} \tag{28}$$

We first see that a probe static in the global time τ cannot be supersymmetric. For such a probe we have $\frac{d\chi}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = \frac{d\psi}{d\tau} = 0$ and the κ-symmetry condition reduces to

$$\begin{aligned}
&\frac{1}{\sqrt{-1-\cos^2 B\sinh^2\chi}} \cdot [(\cosh\chi\cos\tau\cos^2 B + \sin\theta\cos\phi\sin^2 B)\Gamma^0 \\
&+ i\cosh\chi\sin\tau\cos B\Gamma^{01} - i\cos\theta\sin B\Gamma^{02} - i\sin\theta\sin\phi\sin B\Gamma^{03} \\
&+ i(\cosh\chi\cos\tau - \sin\theta\cos\phi)\sin B\cos B\Gamma^{04}]\epsilon_0 = \epsilon_0.
\end{aligned} \tag{29}$$

The terms in this equation proportional to $\cos\tau$, $\sin\tau$ and 1 must all vanish separately, which is clearly impossible. The lack of such configurations is not surprising, because angular momentum must be nonzero for a nontrivial BPS configuration.

Now we allow the probe to orbit around the S^3. Solving the κ-symmetry condition (22) using (28) for Killing spinors obeying

$$\Gamma^{02}\epsilon_0 = \mp\epsilon_0, \tag{30}$$

we find the supersymmetric trajectory at a generic (χ, θ, ψ) to be

$$\frac{d\chi}{d\tau} = \frac{d\theta}{d\tau} = \frac{d\psi}{d\tau} = 0, \qquad \frac{d\phi}{d\tau} = \pm 1. \tag{31}$$

This is a probe orbiting along the ϕ-direction.

The constraint on the Killing spinor (30) projects out half of the components of ϵ_0, i.e. the orbiting zero-brane probe is a half-BPS configuration. We will show in the next subsection, using the BPS bound, that this supersymmetric trajectory is unique up to rotations.

3.1.2 A BPS bound

The worldline action of a zero brane probe, with mass m and the electric charge q, can be written as

$$S = -m\int \sqrt{h}d\sigma^0 + q\int A_{[1]}, \tag{32}$$

where $A_{[1]}$ is the 1-form gauge field (11). We consider supersymmetric probes which have $q = m$.[4]

In global coordinates with $\sigma^0 = \tau$, the Lagrangian of the system is

$$\begin{aligned}\mathcal{L} &= \frac{\sqrt{Q}}{2}\{-m\sqrt{\cosh^2\chi - \dot{\chi}^2 - [\sin B\sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})]^2 - \dot{\theta}^2 - \sin^2\theta\dot{\phi}^2}\\ &\quad + m[\cos B\sinh\chi + \sin B(\dot{\psi} + \cos\theta\dot{\phi})]\}.\end{aligned} \tag{33}$$

The corresponding Hamiltonian is

$$H = \cosh\chi\sqrt{P_\chi^2 + P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2 + (\frac{\sin BP_\psi - \frac{\sqrt{Q}}{2}m}{\cos B})^2} + \sinh\chi(\frac{\sin BP_\psi - \frac{\sqrt{Q}}{2}m}{\cos B}),$$

where the momenta are

$$\begin{aligned}P_\chi &= \frac{m\sqrt{Q}}{2\sqrt{h}}\dot{\chi},\\ P_\theta &= \frac{m\sqrt{Q}}{2\sqrt{h}}\dot{\theta},\\ P_\phi &= \frac{m\sqrt{Q}}{2}\left[\frac{1}{\sqrt{h}}\left(-\cos B\cos\theta[\sin B\sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})] + \sin^2\theta\dot{\phi}\right) + \sin B\cos\theta\right],\\ P_\psi &= \frac{m\sqrt{Q}}{2}\left[\frac{1}{\sqrt{h}}\left(-\cos B[\sin B\sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})]\right) + \sin B\right],\end{aligned} \tag{34}$$

and

$$h = \cosh^2\chi - \dot{\chi}^2 - [\sin B\sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})]^2 - \dot{\theta}^2 - \sin^2\theta\dot{\phi}^2. \tag{35}$$

[4] The zero-brane can be obtained by wrapping M2-branes on the holomorphic two-cycles of the Calabi-Yau threefold X. It carries electric charges v_A, $A = 1, 2, ..b_2(X)$. Then $m = q = \frac{v_A y^A}{\sqrt{Q/2}}$.

The unbroken rotational symmetries lead to the conserved charges:

$$\begin{aligned} J^1_{\text{right}} &= \sin\phi P_\theta + \cos\phi(\cot\theta P_\phi - \csc\theta P_\psi), \\ J^2_{\text{right}} &= \cos\phi P_\theta - \sin\phi(\cot\theta P_\phi - \csc\theta P_\psi), \\ J^3_{\text{right}} &= P_\phi, \\ J^3_{\text{left}} &= P_\psi. \end{aligned} \tag{36}$$

It is easy to see that there are no static solutions. They would have to minimize the potential energy according to

$$0 = \frac{\partial H}{\partial \chi} = \frac{\sqrt{Q}}{2} m \cos B \cosh\chi \left(\frac{\cos B \sinh\chi}{\sqrt{\cos^2 B \sinh^2\chi + 1}} - 1\right), \tag{37}$$

which has no solutions for finite χ. Physically, the probe is accelerated to $\chi = \pm\infty$.

Now we allow the probe to orbit. Solutions of this type can be stabilized by the angular potential. The supersymmetric configuration turns out to be at constant radius in the AdS_2, i.e. $P_\chi = 0$. The Hamiltonian is minimized with respect to χ when

$$\tanh\chi = -\frac{1}{\sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2 + (\frac{\sin B P_\psi - \frac{\sqrt{Q}}{2}m}{\cos B})^2}} \left(\frac{\sin B P_\psi - \frac{\sqrt{Q}}{2} m}{\cos B}\right). \tag{38}$$

The value of H at the minimum is

$$H_{min} = \sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2} = |\vec{J}_{\text{right}}|, \tag{39}$$

where $|\vec{J}_{\text{right}}|^2 = (J^1_{\text{right}})^2 + (J^2_{\text{right}})^2 + (J^3_{\text{right}})^2$. This implies the BPS bound

$$H \geq |\vec{J}_{\text{right}}| \tag{40}$$

for generic χ.

Up to spatial rotations, we may always choose static BPS solutions to satisfy

$$H = J^3_{\text{right}} = \pm P_\phi, \qquad J^1_{\text{right}} = J^2_{\text{right}} = 0. \tag{41}$$

This implies

$$P_\theta = 0, \qquad \cos\theta P_\phi = P_\psi. \tag{42}$$

Hence, the azimuthal angle is determined by the ratio of left and right angular momenta:

$$\cos\theta = \frac{J^3_{\text{left}}}{J^3_{\text{right}}}. \tag{43}$$

We can rewrite $\dot{\phi}$ and $\dot{\psi}$ in terms of P_ϕ and P_ψ. With $\dot{\chi} = \dot{\theta} = 0$,

$$\dot{\phi} = \frac{\cosh\chi(\frac{P_\phi - \cos\theta P_\psi}{\sin^2\theta})}{\sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2 + (\frac{\sin B P_\psi - \frac{\sqrt{Q}}{2}m}{\cos B})^2}}, \tag{44}$$

$$\dot{\psi} = \frac{\cosh\chi[\tan B(\frac{\sin B P_\psi - \frac{\sqrt{Q}}{2}m}{\cos B}) - (\frac{\cos\theta P_\phi - P_\psi}{\sin^2\theta})]}{\sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2 + (\frac{\sin B P_\psi - \frac{\sqrt{Q}}{2}m}{\cos B})^2}} + \tan B \sinh\chi. \tag{45}$$

Eliminate χ through (38),

$$\dot{\phi} = \frac{1}{\sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2}}(\frac{P_\phi - \cos\theta P_\psi}{\sin^2\theta}), \tag{46}$$

$$\dot{\psi} = \frac{1}{\sqrt{P_\theta^2 + (\frac{\cos\theta P_\phi - P_\psi}{\sin\theta})^2 + P_\phi^2}}(\frac{P_\psi - \cos\theta P_\phi}{\sin^2\theta}). \tag{47}$$

Plug in (42), the solution is

$$\dot{\theta} = 0, \qquad \dot{\phi} = \pm 1, \qquad \dot{\psi} = 0, \tag{48}$$

for which (P_ϕ, P_ψ) are

$$P_\psi = \pm\frac{\sqrt{Q}}{2}m\frac{\cos\theta}{\cos B\sinh\chi \pm \sin B\cos\theta}, \tag{49}$$

$$P_\phi = \pm\frac{\sqrt{Q}}{2}m\frac{1}{\cos B\sinh\chi \pm \sin B\cos\theta}. \tag{50}$$

The energy of the particle following this trajectory is equal to $\pm P_\phi$:

$$H = \frac{\sqrt{Q}}{2}m\frac{1}{\cos B\sinh\chi \pm \sin B\cos\theta} = \pm P_\phi. \tag{51}$$

We see that the solution with $\dot{\phi} = 1$ ($\dot{\phi} = -1$) corresponds to a chiral (anti-chiral) BPS configuration.

Therefore, we have confirmed that the supersymmetric trajectories (31) obtained by solving the κ-symmetry condition correspond to the BPS states.

3.1.3 Poincaré coordinates

In Poincaré coordinates and static gauge $\sigma^0 = t$, the κ-symmetry condition for a static probe is

$$\frac{1}{\sqrt{-\frac{1}{\sigma^2}}}\left[-\frac{1}{\sigma}\Gamma^0\right]\epsilon = i\Gamma^0\epsilon = \epsilon. \tag{52}$$

This equation is solved by simply taking $\epsilon = \epsilon^+ = \frac{1}{\sqrt{\sigma}} R(\theta, \phi, \psi)\epsilon_0^+$. Again, we find a half-supersymmetric solution, although the broken supersymmetries are different than in the global case. It can be seen that there are no supersymmetric orbiting trajectories in Poincaré time.

3.2 One-brane probe

In this subsection, we find some supersymmetric one-brane configurations. We consider a specific Ansatz with no worldvolume electromagnetic field and with the one-brane geometry:

$$\begin{aligned} \tau &= \sigma^0, \\ \phi &= \dot{\phi}\sigma^0 + \phi'\sigma^1, \\ \psi &= \dot{\psi}\sigma^0 + \psi'\sigma^1, \end{aligned} \tag{53}$$

where (σ^0, σ^1) are worldvolume coordinates, and $\dot{\phi}$, $\dot{\psi}$, ϕ' and ψ' are all taken to be constant. Note that since (ψ, ϕ) are the orbits of (J_L^3, J_R^3), they may be viewed as one-brane momentum-winding modes on the torus generated by (J_L^3, J_R^3). This torus degenerates to a circle at the loci $\theta = \{0, \pi\}$. One-branes of the form (53) at these loci are therefore static (up to reparametrizations).

With no electromagnetic field the κ-symmetry condition is[5]

$$\frac{1}{2}\epsilon^{ij}\tilde{\Gamma}_{ij}\epsilon = \epsilon, \tag{54}$$

where h and $\tilde{\Gamma}_i$ are the pull-backs of the 5D metric and gamma matrices onto the one-brane worldsheet. With the Ansatz (53), we have explicitly

$$\tilde{\Gamma}_0 = \Gamma_\tau + \dot{\phi}\Gamma_\phi + \dot{\psi}\Gamma_\psi, \tag{55}$$
$$\tilde{\Gamma}_1 = \phi'\Gamma_\phi + \psi'\Gamma_\psi, \tag{56}$$
$$\frac{1}{2}\epsilon^{ij}\tilde{\Gamma}_{ij} = \frac{1}{2\sqrt{\det h}}[\phi'\Gamma_{\tau\phi} + \psi'\Gamma_{\tau\psi} + (\dot{\phi}\psi' - \dot{\psi}\phi')\Gamma_{\phi\psi}], \tag{57}$$

and

$$\begin{aligned} h_{00} &= \frac{Q}{4}\{-\cosh^2\chi + [\sin B \sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})]^2 + \sin^2\theta\ \dot{\phi}^2\}, \\ h_{11} &= \frac{Q}{4}\{\cos^2 B(\psi' + \cos\theta\phi')^2 + \sin^2\theta\ \phi'^2\}, \\ h_{01} &= \frac{Q}{4}\{[\sin B\sinh\chi - \cos B(\dot{\psi} + \cos\theta\dot{\phi})](-\cos B)(\psi' + \cos\theta\phi') + \sin^2\theta\ \dot{\phi}\phi'\}, \end{aligned} \tag{58}$$

[5]There is a simple kappa-symmetric action in six dimensions, but not in five. In 5D we expect an extra scalar field along with the transverse coordinates to fill out the supermultiplet. For the case of the M5-brane wrapping a Calabi-Yau 4-cycle, the scalar in the effective one-brane arises as a mode of the antisymmetric tensor field. The Ansatz of this section corresponds to taking this extra scalar to be a constant.

and hence

$$\det h = (\frac{Q}{4})^2\{\cosh^2\chi[\cos^2 B(\psi'+\cos\theta\phi')^2+\sin^2\theta\phi'^2]-\sin^2\theta[\sin B\sinh\chi\phi'-\cos B(-\psi'\dot{\phi}+\phi'\dot{\psi})]^2\}. \tag{59}$$

It is simplest to analyze the κ-symmetry condition in the form

$$S^{-1}\frac{1}{2}\epsilon^{ij}\tilde{\Gamma}_{ij}S\epsilon_0 = \epsilon_0. \tag{60}$$

The rotated gamma matrices appearing in this expression are explicitly

$$\begin{aligned} & S^{-1}\Gamma_{\tau\phi}S \\ = & -\frac{Q}{4}[(\cosh^2\chi\cos^2 B+\sin^2\theta\sin^2 B)\Gamma^{02} - i(\cosh\chi\cos\tau\cos^2 B+\sin\theta\cos\phi\sin^2 B \\ & -i\cosh\chi\sin\tau\cos B\Gamma^1 + i\sin\theta\sin\phi\sin B\Gamma^3 \\ & -i(\cosh\chi\cos\tau-\sin\theta\cos\phi)\sin B\cos B\Gamma^4)(\cos\theta\sin B\Gamma^0+\sinh\chi\cos B\Gamma^2)], \end{aligned} \tag{61}$$

$$\begin{aligned} & S^{-1}\Gamma_{\tau\psi}S \\ = & \frac{Q}{4}\cos B\{-\cosh^2\chi\cos\theta\cos B\Gamma^{02} \\ & +\cos B\sinh\chi[i\cosh\chi\sin\theta\cos\tau\cos\phi\Gamma^4+\cosh\chi\sin\theta\sin\phi\sin\tau\Gamma^{13} \\ & -\cosh\chi\sin\theta\cos\tau\sin\phi(\sin B\Gamma^{34}-i\cos B\Gamma^3) \\ & +\cosh\chi\sin\theta\cos\phi\sin\tau(\cos B\Gamma^{14}+i\sin B\Gamma^1)] \\ & -(\cos^2 B\cosh^2\chi\sin\theta\cos\phi+\sin^2 B\cosh\chi\cos\tau)\Gamma^{04} - \sin B\cos B\cosh\chi\sinh\chi\cos\tau\cos\theta\Gamma^{24} \\ & -\cosh\chi\sin\tau\sin B\Gamma^{01}+\cos B\cosh\chi\sinh\chi\cos\theta\sin\tau\Gamma^{12} \\ & -\cosh^2\chi\sin\theta\sin\phi\cos B\Gamma^{03} \\ & -i\cosh\chi(\cosh\chi\sin\theta\cos\phi-\cos\tau)\sin B\cos B\Gamma^0 + i\cos^2 B\cosh\chi\sinh\chi\cos\tau\cos\theta\Gamma^2\}, \end{aligned} \tag{62}$$

$$\begin{aligned} & S^{-1}\Gamma_{\phi\psi}S \\ = & \frac{Q}{4}\cos B\{+\sinh\chi\sin^2\theta\sin B\Gamma^{02} \\ & +\sin B\cos\theta[i\cosh\chi\sin\theta\cos\phi\cos\tau\Gamma^4+\cosh\chi\sin\theta\sin\phi\sin\tau\Gamma^{13} \\ & +\cosh\chi\sin\theta\cos\phi\sin\tau(\cos B\Gamma^{14}+i\sin B\Gamma^1) \\ & -\cosh\chi\sin\theta\sin\phi\cos\tau(\sin B\Gamma^{34}-i\cos B\Gamma^3)] \\ & -\sin B\cos B\sinh\chi\sin\theta\cos\theta\cos\phi\Gamma^{04}+(\cosh\chi\sin^2\theta\cos\tau\sin^2 B+\sin\theta\cos\phi\cos^2 B)\Gamma^{24} \\ & -\cosh\chi\sin^2\theta\sin\tau\sin B\Gamma^{12} \\ & -\sin B\sin\theta\cos\theta\sin\phi\sinh\chi\Gamma^{03}+\sin\theta\sin\phi\cos B\Gamma^{23} \\ & -i\sin^2 B\sinh\chi\sin\theta\cos\theta\cos\phi\Gamma^0 - i\sin\theta(\cosh\chi\sin\theta\cos\tau-\cos\phi)\cos B\sin B\Gamma^2\}. \end{aligned} \tag{63}$$

This all simplifies at points obeying

$$\sinh\chi = \pm\tan B\cos\theta \tag{64}$$

when $-\psi'\dot{\phi}+\phi'\dot{\psi}=\pm\psi'$. Under these conditions

$$\sqrt{\det h} = \frac{Q}{4}(\phi'+\cos\theta\psi'), \tag{65}$$

and

$$\begin{aligned} & S^{-1}[\phi'\Gamma_{\tau\phi}+\psi'\Gamma_{\tau\psi}+(\dot{\phi}\psi'-\dot{\psi}\phi')\Gamma_{\phi\psi}]S \\ = \; & \frac{Q}{4}[-(\phi'+\cos\theta\psi')\Gamma^{02}+(\phi'\widehat{D}_1+\psi'\widehat{D}_2)(\Gamma^0\pm\Gamma^2)], \end{aligned} \tag{66}$$

where

$$\begin{aligned} \widehat{D}_1 &= i\cos\theta\sin B[\cosh\chi\cos\tau\cos^2 B+\sin\theta\cos\phi\sin^2 B \\ &\quad -i\cosh\chi\sin\tau\cos B\Gamma^1+i\sin\theta\sin\phi\sin B\Gamma^3-i(\cosh\chi\cos\tau-\sin\theta\cos\phi)\sin B\cos B\Gamma^4], \\ \widehat{D}_2 &= -\cos B(\cos^2 B\sin\theta\cos\phi+\sin^2 B\cosh\chi\cos\tau)\Gamma^4+\cos B\sin B\cosh\chi\sin\tau\Gamma^1 \\ &\quad +\cos^2 B\sin\theta\sin\phi\Gamma^3-i\sin B\cos^2 B(\sin\theta\cos\phi-\cosh\chi\cos\tau). \end{aligned}$$

So far we have not chosen which supersymmetries are to be preserved. We take those generated by spinors obeying $\Gamma^{02}\epsilon_0=\pm\epsilon_0$, or equivalently $\Gamma^2\epsilon_0=\mp\Gamma^0\epsilon_0$. In this case, the last term in (66) can be dropped and the supersymmetry conditions are satisfied.

To summarize, any configuration satisfying

$$\begin{aligned} -\psi'\dot{\phi}+\phi'\dot{\psi} &= \pm\psi', \qquad \dot{\chi}=\dot{\theta}=0, \\ \sinh\chi &= \pm\tan B\cos\theta \end{aligned} \tag{67}$$

preserves those supersymmetries corresponding to

$$\Gamma^{02}\epsilon_0=\pm\epsilon_0. \tag{68}$$

Other BPS configurations preserving other sets of supersymmetries can be obtained by $SL(2,R)\times SO(4)$ rotations of these ones.

Note that, as for the zero-branes, there are generic solutions for any θ. These include $\theta=\{0,\pi\}$, which correspond to static one-branes because the (ψ,ϕ) torus degenerates to a circle along these loci. Static solutions are possible because a one-brane probe in 5D couples magnetically to the dual of the spacetime gauge field $F_{[2]}$ of (11) hence there is nonzero angular momentum carried by the fields.

4 Acknowledgments

We would like to thank M. Ernebjerg, D. Gaiotto, L. Huang, J. Lapan and X. Yin for many helpful discussions. The work was supported by DOE grant DE-FG02-91ER40654.

References

[1] A. Simons, A. Strominger, D. M. Thompson and X. Yin, "Supersymmetric branes in AdS(2) x S**2 x CY(3)," Phys. Rev. D **71**, 066008 (2005) [arXiv:hep-th/0406121].

[2] D. Gaiotto, A. Simons, A. Strominger and X. Yin, "D0-branes in black hole attractors," arXiv:hep-th/0412179.

[3] D. Gaiotto, A. Strominger and X. Yin, "Superconformal black hole quantum mechanics," arXiv:hep-th/0412322.

[4] D. Gaiotto, A. Strominger and X. Yin, arXiv:hep-th/0602046.

[5] H. Ooguri, A. Strominger and C. Vafa, Phys. Rev. D **70**, 106007 (2004) [arXiv:hep-th/0405146].

[6] D. Gaiotto, A. Strominger and X. Yin, "New connections between 4D and 5D black holes," JHEP **0602**, 024 (2006) [arXiv:hep-th/0503217].

[7] J. C. Breckenridge, R. C. Myers, A. W. Peet and C. Vafa, "D-branes and spinning black holes," Phys. Lett. B **391**, 93 (1997) [arXiv:hep-th/9602065].

[8] N. Alonso-Alberca, E. Lozano-Tellechea and T. Ortin, "Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes," Class. Quant. Grav. **19**, 6009 (2002) [arXiv:hep-th/0208158].

[9] N. Alonso-Alberca, E. Lozano-Tellechea and T. Ortin, Class. Quant. Grav. **20**, 423 (2003) [arXiv:hep-th/0209069].

[10] J. P. Gauntlett, R. C. Myers and P. K. Townsend, "Black holes of D = 5 supergravity," Class. Quant. Grav. **16**, 1 (1999) [arXiv:hep-th/9810204].

[11] K. Becker, M. Becker and A. Strominger, "Five-branes, membranes and nonperturbative string theory," Nucl. Phys. B **456**, 130 (1995) [arXiv:hep-th/9507158].

[12] E. Bergshoeff, R. Kallosh, T. Ortin and G. Papadopoulos, "Kappa-symmetry, supersymmetry and intersecting branes," Nucl. Phys. B **502**, 149 (1997) [arXiv:hep-th/9705040].